École Polytechnique Fédérale de Lausanne

On the subadditive ergodic theorem and its applications

Bachelor Thesis

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Abstract

This bachelor thesis focuses on Ergodic Theory and aims to investigate important mathematical tools and concepts within the field. Our primary objective is to state and prove the Subadditive Ergodic Theorem, which will serve as a stepping stone to derive the Birkhoff Ergodic Theorem and Furstenberg's Theorem. Furthermore, we aim to introduce the concept of Lyapunov Exponents through Oseledets' Multiplicative Ergodic Theorem.

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CHAPTER 1

INTRODUCTION

In the 19th century, Boltzmann was working on the kinetic theory of gases. While trying to formalize in mathematical terms the behaviour of gases, Boltzmann made a hypothesis that typical orbits of Hamiltonian flows fill in the whole energy surface that contains them. Starting from this *ergodic hypothesis*, he inferred that time averages of observable quantities along typical orbits would coincide with space averages of those quantities on the energy surface. However, this hypothesis was later found to be false, leading to the distinction of systems as either *ergodic* or non-ergodic. The exploration of whether the majority of Hamiltonian systems, especially those relevant to the kinetic theory of gases, are ergodic or not, motivated the development of ergodic theory in the 20th century. During this period, significant contributions were made by mathematicians such as Birkhoff and von Neumann. Birkhoff's Ergodic Theorem states that time averages of integrable functions along the orbits of a measure-preserving system are well-defined for almost every point and establishes a connection between time averages along orbits and space averages over the entire phase space. While Birkhoff's Theorem focuses on additive sequences as the object used to describe time averages along orbits, the Subadditive Ergodic Theorem serves as a natural generalization, dealing with subadditive sequences instead.

In this project, our aim is to present and prove Kingman's Subadditive Ergodic Theorem and then introduce the Lyapunov Exponents through Oseledets'Theorem which provides the theoretical background for computation of Lyapunov exponents. Lyapunov exponents are used to quantify the rate of exponential divergence or convergence of nearby trajectories in a dynamical system. They measure the sensitivity of the system to initial conditions and provide information about its stability and chaotic behavior. Positive Lyapunov exponents indicate chaotic dynamics, while negative exponents imply stability and convergence.

Structure

- Chapter 2 introduces some basic notions - Measure Preserving Systems, Ergodic Systems, the Birkhof Ergodic Theorem - and illustrate those notions with the example of the torus.
- Chapter 3 focuses on the Subadditive Ergodic Theorem and provides a proof based on the work of Avila and Bochi [3].
- Chapter 4 deals with the applications of the Subadditive Ergodic Theorem. We prove Furstenberg's Theorem, state Oseledets Theorem and prove it in dimension 2.

CHAPTER 2

PRELIMINARIES

2.1 MEASURE PRESERVING SYSTEMS

Definition 2.1.1 (Measure Preserving Transformation). *Given a probability space $(X, \mathcal{A}, \mu)$, we say that a measurable map $T : X \to X$ is a measure preserving transformation, or that T preserves the measure μ, or that μ is invariant under T, if for every $A \in \mathcal{A}$ we have $\mu\left(T^{-1}A\right) = \mu(A)$.*

In other words, $T : X \to X$ is a measure preserving transformation if $T_*(\mu) = \mu$ where $T_*(\mu)$ is the push-forward of μ under T.

The push-forward of μ under a measurable map T is the probability measure defined via

$$T_*(\mu)(A) = \mu\left(T^{-1}A\right), \quad \forall A \in \mathcal{A}.$$

The following is a useful characterization of measure preserving transformations.

Proposition 2.1.2. *Let $(X, \mathcal{A}, \mu)$ be a probability space and $T : X \to X$ a measurable map. Then the map T is a measure preserving transformation if and only if*

$$\int \phi d\mu = \int \phi \circ T d\mu$$

for every μ-integrable function $\phi : X \to \mathbb{R}$.

A proof of Proposition 2.1.2 can be found in Proposition 1.1.1 from [2].

Definition 2.1.3 (Measure Preserving System). *A measure preserving system is a quadruple $(X, \mathcal{A}, \mu, T)$ where $(X, \mathcal{A}, \mu)$ is a probability space and $T : X \to X$ is a measure preserving transformation.*

The following lemma will be used in the proof of Theorem 3.1.3.

Lemma 2.1.4. *If $(X, \mathcal{A}, \mu, T)$ is a measure preserving system, then so is $(X, \mathcal{A}, \mu, T^k)$, for any $k \geq 1$.*

Proof. Let $k \in \mathbb{N}$. Since μ is invariant under T, for any $A \in \mathcal{A}$ we have $\mu(T^{-1}(A)) = \mu(A)$. T being measurable, $T^{-i}(A) \in \mathcal{A}$ for all $i \in \mathbb{N}$ and for all $A \in \mathcal{A}$. Therefore $\mu(A) = \mu(T^{-1}(A)) = \mu(T^{-2}(A)) = ... = \mu(T^{-k}(A))$ for all $A \in \mathcal{A}$. This shows that T^k preserves μ and ends the proof. $\square$

2.2 BIRKHOFF'S ERGODIC THEOREM

Birkhoff's Ergodic Theorem, as mentioned earlier, states that time averages of integrable functions along the orbits are well-defined for almost every point and provides a connection between the long-term behavior of a system and its statistical properties.

Theorem 2.2.1 (Birkhoff)**.** *Consider a measure preserving system $(X, \mathcal{A}, \mu, T)$. Given any integrable function $\varphi : X \to \mathbb{R}$, the limit*

$$\tilde{\varphi}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j(x)) \tag{2.1}$$

exists at μ-almost every point $x \in X$. Moreover, the function $\tilde{\varphi}$ defined in this way is integrable and satisfies

$$\int \tilde{\varphi}(x) d\mu(x) = \int \varphi(x) d\mu(x).$$

Birkhoff's theorem will be deduced from Kingman's subadditive ergodic theorem, see Theorem 3.1.3.

Lemma 2.2.2. *The function $\tilde{\varphi}$ as defined in* (2.1)*, is an a.e. invariant function.*

Proof. Indeed, we can write

$$\begin{aligned} \tilde{\varphi}(T(x)) &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \varphi(T^j(x)) \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j(x)) + \frac{1}{n}(\varphi(T^n(x)) - \varphi(x)) \\ &= \tilde{\varphi}(x) + \lim_{n\to\infty} \frac{1}{n} \varphi(T^n(x)) \end{aligned}$$

and we conclude by Lemma 2.3.3. □

The following lemma will also serves us later on in the proof of Kingman's Theorem.

Lemma 2.2.3. *Let $\varphi : X \to \mathbb{R}$ be an integrable function. Then $\lim_{n\to\infty} \frac{1}{n}\varphi(T^n(x)) = 0$ for μ-a.e. point $x \in X$.*

Proof. Fix any $\varepsilon > 0$. For any $n \in \mathbb{N}$,

$$\begin{aligned} \mu\left(\{x \in X : |\phi\left(T^n(x)\right)| \geq n\varepsilon\}\right) &= \mu(\{x \in X : |\phi(x)| \geq n\varepsilon\}) \\ &= \sum_{k=n}^{\infty} \mu\left(\left\{x \in X : k \leq \frac{|\phi(x)|}{\varepsilon} < k+1\right\}\right). \end{aligned}$$

Summing these over $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \mu\left(\{x \in X : |\phi\left(T^n(x)\right)| \geq n\varepsilon\}\right) &= \sum_{k=1}^{\infty} k\mu\left(\left\{x \in X : k \leq \frac{|\phi(x)|}{\varepsilon} < k+1\right\}\right) \\ &\leq \int \frac{|\phi|}{\varepsilon} d\mu. \end{aligned}$$

All these expressions are finite, since ϕ is integrable by assumption. By the Borel-Cantelli lemma, the set $B(\varepsilon)$ of all points x such that $|\phi(T^n(x))| \geq n\varepsilon$ for infinitely many values of n has zero measure. Now, the definition of $B(\varepsilon)$ implies that for every $x \notin B(\varepsilon)$ there exists $p \geq 1$ such that $|\phi(T^n(x))| < n\varepsilon$ for every $n \geq p$. Let $B = \bigcup_{i=1}^{\infty} B(1/i)$. Then B has zero measure and $\lim_n (1/n)\phi(T^n(x)) = 0$ for every $x \notin B$. □

In the next section, we will introduce the concept of ergodicity, which will enable us to strengthen the Birkhoff Ergodic Theorem in the case where the system under consideration is ergodic.

2.3 ERGODICITY

A measure preserving system is said to be ergodic if it is dynamically indivisible, meaning that every invariant set has either full measure or zero measure. Proposition 2.3.5 shows that ergodicity also means that the amount of time an orbit $x, T(x), T^2(x), \ldots$ of a typical point $x \in X$ spends in a given measurable set is proportional to the measure of that set. We introduce the Mean Sojourn time $\tau(A, x)$ below to express this property.

Definition 2.3.1 (Ergodic System)**.** *A measure preserving system $(X, \mathcal{A}, \mu, T)$ is said to be ergodic, if for every set $A \in \mathcal{A}$,*

$$T^{-1}A = A \Longrightarrow \mu(A) = 0 \text{ or } \mu(A) = 1.$$

Whenever $T^{-1}A = A$, the set A is called strictly invariant, and it is almost everywhere invariant if $\mu\left(A \triangle T^{-1}A\right) = 0$.

We also call a measurable function $\varphi : X \to \mathbb{R}$ strictly invariant if $\varphi \circ T = \varphi$ and almost everywhere invariant if $\varphi \circ T = \varphi$ almost surely.

The next proposition gives some equivalent definitions of ergodicity. A proof can be found in Proposition 7 from [1].

Proposition 2.3.2. *Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving system. The following are equivalent:*

(i) *$(X, \mathcal{A}, \mu, T)$ is ergodic.*

(ii) *If $A \in \mathcal{A}$ is almost everywhere invariant then either $\mu(A) = 0$ or $\mu(A) = 1$.*

(iii) *If $\varphi : X \to \mathbb{R}$ is measurable and strictly invariant then φ is equal to a constant almost everywhere.*

(iv) *If $\varphi : X \to \mathbb{R}$ is measurable and almost everywhere invariant then φ is equal to a constant almost everywhere.*

Definition 2.3.3 (Mean Sojourn Time)**.** *Consider a measure preserving system $(X, \mathcal{A}, \mu, T)$. Given a measurable set $A \in \mathcal{A}$ and a point $x \in X$, we define the mean sojourn time of x at A via*

$$\tau(A, x) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_A\left(T^j(x)\right)$$

whenever the limit exists, with χ_A being the characteristic function of the set A.

Proposition 2.3.4 (Existence of the mean sojourn time)**.** *Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving system, then $\tau(A, x)$ exists at μ-almost every point $x \in X$, for every measurable set A. Moreover, $\int \tau(A, x) d\mu = \mu(A)$.*

Proof. This is a direct consequence of Birkhoff's ergodic theorem, see Theorem 2.2.1. □

Proposition 2.3.5. *Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving system. The following are equivalent:*

(i) $(X, \mathcal{A}, \mu, T)$ *is ergodic.*

(ii) $\tau(A, x) = \mu(A)$ *for μ-almost every $x \in X$ and for every $A \subseteq X$ measurable.*

Proof. To prove that (i) implies (ii), we notice from Lemma 2.2.2 that $\tau(A, \cdots)$ is an invariant fumction, then we deduce from Proposition 2.3.2-(iv) that it is a.s. constant, and finally we conclude from Proposition 2.3.4 that it coincides with $\mu(A)$. To prove that (ii) implies (i), let us consider a strictly invariant measurable set A. Then $\tau(x, A) = \chi_A(x)$, since $\tau(x, A) = \mu(A)$ for μ almost every point, then $\mu(A) = 0$ or $\mu(A) = 1$ and we are done. □

Remark 2.3.6. If the measure preserving system $(X, \mathcal{A}, \mu, T)$ is ergodic, Birkhoff's Theorem can be strengthened. Indeed, Lemma 2.2.2 tells us that $\tilde{\varphi}$ is an almost everywhere invariant function, while Proposition 2.3.2 implies that $\tilde{\varphi}$ must be a equal to a constant almost everywhere. Therefore we may write that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j(x)) = \int \varphi d\mu,$$

for μ-a.e. $x \in X$.

2.4 EXAMPLES

In this section, we illustrate the concepts discussed above using the example of the torus $\mathbb{T}$, which is identified to the segment $[0, 1]$ with periodic boundary condition, equipped with the Lebesgue measure and the Borel σ-algebra.

Let $R_\alpha : x \mapsto x + \alpha$ mod 1 be the rotation by α on the torus. R_α leaves the Lebesgue measure invariant, so that $(\mathbb{T}, \mathcal{B}, \mathcal{L}, R_\alpha)$ is a measure preserving system. We are interested in understanding when this system is ergodic or not.

Proposition 2.4.1. *If $\alpha \in \mathbb{Q}$, then $(\mathbb{T}, \mathcal{B}, \mathcal{L}, R_\alpha)$ is not an ergodic system.*

Proof. Let us write $\alpha = \frac{p}{q}$ in his irreducible form. Consider the following set:

$$A := \bigcup_{k=0}^{q-1} \left[k\alpha - \frac{1}{4q}, k\alpha + \frac{1}{4q}\right]$$

By construction, A is a strictly invariant set which has Lebesgue measure $\frac{1}{2}$. Therefore the system cannot be ergodic when α is rational. □

In fact, it can be shown that the system is ergodic if, and only if, α is an irrational number. This result follows from Weyl's Equidistribution Theorem, which we state below.

Lemma 2.4.2. *For $\varphi \in C^0([0, 1])$ 1-periodic function and α an irrational number. The following holds*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(j\alpha) \to \int_0^1 \varphi(x)dx.$$

The proof of Lemma 2.4.2 is based on Fourier analysis. The idea is to show the result for functions $\varphi(x) = e^{2\pi ikx}$ and then use Fejer's Theorem to conclude. A detailed proof can be found at pp. 499-501 from [5].

Theorem 2.4.3 (Weyl's Equidistribution)**.** *For any α irrational and $0 \leq a < b < 1$ the following holds*

$$\tau((a,b),\alpha) = \lim_{n\to\infty} \frac{1}{n}\#\{j \in \{1,...,n\} : \{j\alpha\} \in (a,b)\} = b - a$$

Weyl's Equidistribution Theorem is a consequence of Lemma 2.4.2. A detailed proof can be found at pp. 499-501 from [5].

Now we observe that the left-hand side in Weyl's theorem corresponds to the sojourn time $\tau((a,b),\alpha)$, while the right-hand side represents the measure of the interval (a,b). In fact, the result holds not only for intervals, but can be extented to all Borel sets and therefore, applying Proposition 2.3.5, we conclude that $(\mathbb{T}, \mathcal{B}, \mathcal{L}, R_\alpha)$ is ergodic when α is irrational.

Remark 2.4.4. The same ideas can be extended to the two-dimensional torus. For any pair $(\alpha_1, \alpha_2) \in [0,1)^2$, we can consider translations on the torus $\mathbb{T}^2$ given by:

$$R_{(\alpha_1,\alpha_2)}(x,y) = (x + \alpha_1, y + \alpha_2) \quad \forall (x,y) \in \mathbb{T}^2.$$

It can be demonstrated that $(\mathbb{T}^2, \mathcal{B}, \mathcal{L}^2, R_{\alpha,\beta})$ forms an ergodic measure preserving system if, and only if, α and β are irrational numbers and there does not exist $k_1, k_2 \in \mathbb{Z}$ such that $k_1\alpha + k_2\beta \in \mathbb{Z}$.

CHAPTER 3

KINGMAN'S SUBADDITIVE ERGODIC THEOREM

3.1 THE SUBADDITIVE ERGODIC THEOREM

In what follows, we always consider $(X, \mathcal{A}, \mu, T)$ to be a measure preserving system.

Definition 3.1.1 (Subadditive sequence of functions)**.** *A sequence of measurable functions $\varphi_n : X \to [-\infty, +\infty)$, $n \in \mathbb{N}$, is said to be subadditive with respect to T if for all $m, n \geq 1$, we have the following*

$$\varphi_{m+n} \leq \varphi_m + \varphi_n \circ T^m.$$

Notice that if $(\varphi_n)_n$ is subadditive, then for any $n \in \mathbb{N}$ it holds

$$\varphi_n \leq \sum_{j=0}^{n-1} \varphi_1 \circ T^j. \tag{3.1}$$

Definition 3.1.2 (Additive sequence of functions)**.** *A sequence of measurable functions $\varphi_n : X \to [-\infty, +\infty)$, $n \in \mathbb{N}$, is said to be subadditive with respect to T if for all $m, n \geq 1$, we have the following*

$$\varphi_{m+n} = \varphi_m + \varphi_n \circ T^m.$$

In fact, we observe that $(\varphi_n)_n$ is an additive sequence if, and only if, it can be written as

$$\varphi_n = \sum_{j=0}^{n-1} \varphi_1 \circ T^j \tag{3.2}$$

for any $n \in \mathbb{N}$.

While Birkhoff's theorem deals with additive sequences, which are the objects used to describe time averages along orbits, Kingman's theorem generalizes this concept to subadditive sequences, providing convergence results for their averages.

The positive part of a measurable function $\varphi : X \to [-\infty, +\infty]$ is defined as $\varphi^+(x) = \max\{0, \varphi(x)\}$. Similarly, the negative part is $\varphi^-(x) = \max\{0, -\varphi(x)\}$.

Theorem 3.1.3 (Kingman)**.** *Consider a subadditive sequence of measurable functions* $\varphi_n : X \to [-\infty, +\infty)$*, such that the positive part* φ_1^+ *of* φ_1 *is an integrable function. Then* $(\varphi_n/n)_n$ *converges* μ*-almost everywhere to some invariant function* $\varphi : X \to [-\infty, +\infty)$ *whose positive part* φ^+ *is integrable and*

$$\int \varphi d\mu = \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu = \inf_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu \in [-\infty, +\infty). \tag{3.3}$$

Remark 3.1.4. If we assume that the measure preserving system $(X, \mathcal{A}, \mu, T)$ is ergodic, then Kingman's theorem can be further strengthened so that the equation (3.3) becomes

$$\lim_{n\to\infty} \frac{1}{n}\varphi_n(x) = \lim_{n\to\infty} \frac{1}{n} \int \varphi_n d\mu \text{ for } \mu\text{-a.e. } x \in X.$$

Remark 3.1.5. The sequence $(\sum_{j=0}^{n-1} \varphi \circ T^j)_n$, in Birkhoff's theorem is additive. Therefore, the conclusion of Birkhoff's theorem follows directly from Theorem 3.1.3.

3.2 PROOF OF THE SUBADDITIVE ERGODIC THEOREM

3.2.1 PREPARING THE PROOF

First of all, we show that $\lim \frac{1}{n} \int \varphi_n d\mu = \inf \frac{1}{n} \int \varphi_n d\mu = L \in [-\infty, \infty)$ exists.

Definition 3.2.1 (Subadditive sequence)**.** *A sequence* $(a_n)_n$ *in* $[-\infty, +\infty)$ *is subadditive if, for any* $m, n \in \mathbb{N}$*, we have* $a_{m+n} \le a_m + a_n$ *.*

Notice that inequality (3.1) remains true if we replace φ_n and φ_1 by φ_n^+ and φ_1^+. Since φ_1^+ is integrable, then so is φ_n^+ (using Lemma 2.1.4) and $(\int \varphi_n d\mu)_n$ is a subadditive sequence in $[-\infty, +\infty)$.

Lemma 3.2.2. *If* $(a_n)_n$ *is subadditive, then*

$$\lim \frac{a_n}{n} = \inf \frac{a_n}{n} \in [-\infty, \infty).$$

Proof. If $a_m = -\infty$ for some m, it follows by subadditivity that $a_n = -\infty$, for all $n > m$ and we are done. Otherwise, consider $L = \inf(a_n/n) \in [-\infty, +\infty)$, fix $\epsilon > 0$ and let $k \in \mathbb{N}$ such that $a_k/k < L + \epsilon$. We may write $n = kp + q$, where p and q are integer numbers such that $p \ge 1$ and $1 \le q \le k$. By subadditivity we have

$$\frac{a_n}{n} \le \frac{pk}{n}\frac{a_k}{k} + \frac{a_q}{n}.$$

Since a_q is bounded by $\max_{i=1,..,k}\{a_i\} < \infty$, the right hand side converges to a_k/k when $n \to \infty$. Letting $\epsilon \to \infty$ we conclude. □

It remains to show that $(\varphi_n/n)_n$ converges μ-almost everywhere to some invariant function $\varphi : X \to [-\infty, +\infty)$. Let us introduce the functions $\varphi_-(x) := \liminf_n \frac{\varphi_n}{n}(x)$ and $\varphi_+(x) := \limsup_n \frac{\varphi_n}{n}(x)$, for $x \in X$. Our goal is to prove the following inequalities

$$\int \varphi_- d\mu \ge L \ge \int \varphi_+ d\mu. \tag{3.4}$$

By definition, we have:

$$0 \le \varphi_+(x) - \varphi_-(x) \quad \text{for every } x \in X. \tag{3.5}$$

By combining equations (3.4) and (3.5), we can conclude that φ_- and φ_+ coincide at μ-almost every point, and that their integrals are equal to L, and the theorem would be proved. In fact, the first inequality in (3.4) is the key one, and the second will be obtained as a consequence.

So, in order to prove (3.4), we proceed as follows : In a first time, we make the assumption that there exists a constant C such that $\varphi_n(x) \geq -Cn$ for all $x \in X$ and all $n \in \mathbb{N}$. Under this assumption, we will bound φ_- from below, and bound φ_+ from above. At the end only, we remove the assumption we made, using a truncation trick.

So from now on, until the very end of the proof, we will assume that there exists a constant C such that

$$\varphi_n(x) \geq -Cn \text{ for all } x \in X \text{ and all } n \in \mathbb{N}. \tag{3.6}$$

We also verify in the next lemma that φ (which coincides with φ_- and φ_+ at μ-almost every point by construction) is indeed an invariant function.

Lemma 3.2.3. *φ_- is an almost everywhere invariant function.*

Proof. By subadditivity, we have for every point $x \in X$ that

$$\varphi_-(T(x)) = \liminf \frac{\varphi_n}{n}(T(x)) \geq \liminf \frac{\varphi_{n+1} - \varphi_1}{n}(x) = \varphi_-(x). \tag{3.7}$$

Integrating $\varphi_- \circ T - \varphi_-$ we obtain, by Proposition 2.1.2 that $\varphi_- \circ T = \varphi_-$ a.e. □

3.2.2 BOUNDING φ_- FROM BELOW

For each $k \in \mathbb{N}$, we set

$$E_k = \{x \in X : \varphi_j(x) \leq j(\varphi_-(x) + \varepsilon) \text{ for some } j \in \{1, \dots, k\}\}.$$

Clearly $E_k \subset E_{k+1}$ for any k. We observe also that $X = \bigcup_k E_k$.

We consider the following function

$$\psi_k(x) = \begin{cases} \varphi_-(x) + \varepsilon & \text{if } x \in E_k \\ \varphi_1(x) & \text{if } x \in E_k^c \end{cases}$$

By definition of E_k, we have that $\varphi_1(x) > \varphi_-(x) + \varepsilon$ for every $x \in E_k^c$. Therefore $\psi_k(x)$ decreases to $\varphi_-(x) + \varepsilon$ as $k \to \infty$, for every $x \in X$. In particular, by the monotone convergence theorem,

$$\int \psi_k d\mu \to \int (\varphi_- + \varepsilon) d\mu \quad \text{as } k \to \infty$$

Lemma 3.2.4. *$L \leq \int \varphi_- d\mu$.*

Proof. Let $x \in X$ such that $\varphi_-(x) = \varphi_-(T^j(x))$ for any $j \geq 1$ (φ_- is a.e. invariant function by Lemma 3.2.3). We define inductively a sequence, possibly finite, of integer numbers

$$m_0 \leq n_1 < m_1 \leq n_2 < m_2 < \cdots$$

Let $m_0 = 0$. Given $j \geq 1$, let n_j be the smallest integer greater or equal than m_{j-1} that satisfies $T^{n_j}(x) \in E_k$, if it exists. By definition of E_k, we can find m_j that verifies $1 \leq m_j - n_j \leq k$ and

$$\varphi_{m_j - n_j}\left(T^{n_j}(x)\right) \leq (m_j - n_j)\left(\varphi_-\left(T^{n_j}(x)\right) + \varepsilon\right) \tag{3.8}$$

Now fix $n \geq k$ and consider the largest $l \geq 0$ such that $m_l \leq n$. By subadditivity,

$$\varphi_{n_j - m_{j-1}}\left(T^{m_{j-1}}(x)\right) \leq \sum_{i=m_{j-1}}^{n_j - 1} \varphi_1\left(T^i(x)\right)$$

for any $j = 1, \ldots, l$ such that $m_{j-1} \neq n_j$, and similarly for $\varphi_{n-m_l}\left(T^{m_l}(x)\right)$. So, we may write

$$\varphi_n(x) \leq \sum_{i \in I} \varphi_1\left(T^i(x)\right) + \sum_{j=1}^{l} \varphi_{m_j - n_j}\left(T^{n_j}(x)\right) \tag{3.9}$$

where $I = \bigcup_{j=1}^{l} [m_{j-1}, n_j) \cup [m_l, n)$. Notice that

$$\varphi_1\left(T^i(x)\right) = \psi_k\left(T^i(x)\right) \quad \text{ for any } \quad i \in \bigcup_{j=1}^{l} [m_{j-1}, n_j) \cup [m_l, \min\{n_{l+1}, n\}),$$

since $T^i(x) \in E_k^c$ in all these cases. Moreover, as φ_- is a.e. invariant and $\psi_k \geq \varphi_- + \varepsilon$, the relation (3.8) implies that

$$\varphi_{m_j - n_j}\left(T^{n_j}(x)\right) \leq \sum_{i=n_j}^{m_j - 1} \left(\varphi_-\left(T^i(x)\right) + \varepsilon\right) \leq \sum_{i=n_j}^{m_j - 1} \psi_k\left(T^i(x)\right)$$

for every $j = 1, \ldots, l$. Using (3.9) we conclude that

$$\varphi_n(x) \leq \sum_{i=0}^{\min\{n_{l+1}, n\} - 1} \psi_k\left(T^i(x)\right) + \sum_{i=n_{l+1}}^{n-1} \varphi_1\left(T^i(x)\right).$$

Since $n_{l+1} > n - k$, the following holds for any $n > k \geq 1$ and μ-almost every point $x \in X$

$$\varphi_n(x) \leq \sum_{i=0}^{n-k-1} \psi_k(T^i(x)) + \sum_{i=n-k}^{n-1} \max\{\psi_k, \varphi_1\}(T^i(x)) \tag{3.10}$$

and therefore

$$\frac{1}{n} \int \varphi_n d\mu \leq \frac{n-k}{n} \int \psi_k d\mu + \frac{k}{n} \int \max\{\psi_k, \varphi_1\} d\mu \tag{3.11}$$

Now, since $\max\{\psi_k, \varphi_1\} \leq \max\left\{\varphi_- + \varepsilon, \varphi_1^+\right\}$ and this last function is integrable, the $\limsup_n$ of the right hand side term in (3.11) is less or equal to zero, and taking the limit when $n \to \infty$ we conclude that $L \leq \int \psi_k d\mu$ for any k. Taking the limit when $k \to \infty$ we get

$$L \leq \int \varphi_- d\mu + \varepsilon$$

and making $\varepsilon \to 0$ we conclude. □

Remark 3.2.5. Inequality (3.10) means that the sequence $(\varphi_n)_n$ is bounded from above by an additive sequence and an "error term" that is negligible because it is the sum of a fixed number k of integrable functions, and n may be taken to be much larger than k.

3.2.3 BOUNDING φ_+ FROM ABOVE

Lemma 3.2.6. *For any fixed k,*

$$\limsup_n \frac{\varphi_{kn}}{n} = k \limsup_n \frac{\varphi_n}{n}$$

Proof. The inequality $\leq$ is clear, since φ_{kn}/kn is a subsequence of φ_n/n. To get the other inequality, write $n = kq_n + r_n$ with $r_n \in \{1, \ldots, k\}$. By subadditivity,

$$\varphi_n \leq \varphi_{kq_n} + \varphi_{r_n} \circ T^{kq_n} \leq \varphi_{kq_n} + \psi \circ T^{kq_n} \tag{3.12}$$

where $\psi = \max\left\{\varphi_1^+, \ldots, \varphi_k^+\right\}$. Note that $n/q_n \to k$ as $n \to \infty$ and $\lim_n \psi \circ T^n/n = 0$ at μ-almost every point (by Lemma 2.2.3). Then, dividing (3.12) by n and taking the $\limsup$ when $n \to \infty$, we obtain that

$$\limsup_n \frac{1}{n}\varphi_n \leq \limsup_n \frac{1}{n}\varphi_{kq_n} + \limsup_n \frac{1}{n}\psi \circ T^{kq_n} = \frac{1}{k}\limsup_q \frac{1}{q}\varphi_{kq}.$$

□

Lemma 3.2.7. $\int \varphi_+ d\mu \leq L$.

Proof. For each fixed k and $n \geq 1$, consider $\theta_n = -\sum_{j=0}^{n-1} \varphi_k \circ T^{jk}$. Observe that, for any $n \geq 1$

$$\int \theta_n d\mu = -n \int \varphi_k d\mu, \tag{3.13}$$

where we used Lemma 2.1.4. As the sequence $(\varphi_n)_n$ is subadditive, we have that $\theta_n \leq -\varphi_{kn}$ for any n. Then, using Lemma 3.2.6 ,

$$\theta_- = \liminf_n \frac{\theta_n}{n} \leq -\limsup_n \frac{\varphi_{kn}}{n} = -k \limsup_n \frac{\varphi_n}{n} = -k\varphi_+$$

and so

$$\int \theta_- d\mu \leq -k \int \varphi_+ d\mu. \tag{3.14}$$

Observe also that the sequence $(\theta_n)_n$ is additive: $\theta_{m+n} = \theta_m + \theta_n \circ T^{km}$ for any $m, n \geq 1$. As $\theta_1 = -\varphi_k$ is bounded from above by $-\inf_x \varphi_k(x)$, we also have that the function θ_1^+ is bounded and, consequently, integrable. Then, we can apply Lemma 3.2.4, together with (3.13), to conclude that

$$\int \theta_- d\mu \geq \lim_n \int \frac{\theta_n}{n} d\mu = -\int \varphi_k d\mu. \tag{3.15}$$

Putting the relations (3.14) and (3.15) together, we obtain that

$$\int \varphi_+ d\mu \leq \frac{1}{k} \int \varphi_k d\mu.$$

Finally, we take the infimum on k to obtain $\int \varphi_+ d\mu \leq L$. □

Now, in order to remove the assumption (3.6), and prove the theorem in the general case, we use a truncation trick. Define, for $\kappa > 0$ and $n \geq 1$

$$\varphi_n^\kappa := \max\{\varphi_n, -\kappa n\}$$

$$\varphi_-^\kappa := \max\{\varphi_-, -\kappa\}$$
$$\varphi_+^\kappa := \max\{\varphi_+, -\kappa\}$$

Notice that we may exchange $\liminf_n$ with the function $\max\{\cdot, -\kappa\}$ which is continuous and monotone increasing. Therefore, we have

$$\varphi_-^\kappa = \max\left\{\liminf_n \frac{\varphi_n}{n}, -\kappa\right\} = \liminf_n \max\left\{\frac{\varphi_n}{n}, -\kappa\right\} = \liminf_n (1/n)\varphi_n^\kappa,$$

and similarly,

$$\varphi_+^\kappa = \limsup_n (1/n)\varphi_n^\kappa.$$

Lemma 3.2.2 implies then that

$$\int \varphi_-^\kappa d\mu = \inf_n \frac{1}{n}\int \varphi_n^\kappa d\mu$$

By the monotone convergence theorem, the following holds

$$\int \varphi_n d\mu = \inf_\kappa \int \varphi_n^\kappa d\mu \quad \text{and} \quad \int \varphi_- d\mu = \inf_\kappa \int \varphi_-^\kappa d\mu.$$

Therefore

$$\int \varphi_- d\mu = \inf_\kappa \int \varphi_-^\kappa = \inf_\kappa \inf_n \frac{1}{n}\int \varphi_n^\kappa d\mu = \inf_n \frac{1}{n}\int \varphi_n d\mu = L,$$

which extends the result of Lemma 3.2.4 to the general case.

Now, for any fixed $\kappa > 0$ we know that $\varphi_+^\kappa = \varphi_-^\kappa$ and that this is a T-invariant function. Since $\varphi_-^\kappa \to \varphi_-$ and $\varphi_+^\kappa \to \varphi_+$ whenever $\kappa \to \infty$, it follows by uniqueness of the limit that $\varphi_- = \varphi_+$at μ-almost every point and the proof is complete.

CHAPTER 4

APPLICATIONS

4.1 LYAPUNOV EXPONENTS

The Lyapunov exponent, named after Aleksandr Lyapunov, is a fundamental concept in mathematics used to measure the rate of divergence of closely neighboring trajectories in a dynamical system. It characterizes the separation between trajectories in phase space and is denoted by λ. The maximal Lyapunov exponent (MLE) is the largest exponent among the spectrum of Lyapunov exponents associated with the system. It plays a crucial role in determining the system's predictability and chaotic nature. A positive MLE indicates chaos, particularly under certain conditions such as phase space compactness. While there are multiple Lyapunov exponents corresponding to different orientations of the initial separation vector, the MLE dominates the behavior of trajectories due to its exponential growth. Lyapunov exponents provide a quantitative measure of the system's sensitivity to initial conditions and offer insights into stability, convergence, and chaotic behavior.

We present below the theorem of Furstenberg, which stands as another important consequence of the subadditive ergodic theorem. Let $\mathrm{GL}(d)$ be the set of $d \times d$ invertible matrices, together with the operation of ordinary matrix multiplication. Let also $\mathrm{SL}(d)$ be the set of $d \times d$ matrices having determinant 1. Additionally, let $\|\cdot\|$ denote the Euclidean norm on matrices.

Definition 4.1.1 (cocycle)**.** *Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving system. Consider any measurable function $\theta : X \to \mathrm{GL}(d)$ with values in the group $\mathrm{GL}(d)$. The cocycle defined by θ over T is the sequence of functions defined by*

$$\phi^n(x) = \theta\left(T^{n-1}(x)\right) \cdots \theta(T(x))\theta(x) \text{ for } n \geq 1 \quad \text{and} \quad \phi^0(x) = \mathrm{id}$$

for every $x \in X$.

Remark 4.1.2. The sequence $(\phi^n)_n$ verifies $\phi^{m+n} = \phi^n \circ T^m \cdot \phi^m$ for every $m, n \in \mathbb{N}$. It is also easy to check that, conversely, any sequence $(\phi^n)_n$ with this property is the cocycle defined by $\theta = \phi^1$ over the transformation T.

Theorem 4.1.3 (Furstenberg)**.** *Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving system. Given any measurable function $\theta : X \to \mathrm{GL}(d)$, if $\log^+ \|\theta\| \in L^1(\mu)$ then*

$$\lambda_{\max}(x) = \lim_n \frac{1}{n} \log \|\phi^n(x)\|$$

exist for μ-almost everywhere $x \in X$. Moreover, $\lambda_{\max}^+$ is μ-integrable, with

$$\int \lambda_{\max} d\mu = \lim_n \frac{1}{n} \int \log \|\phi^n\| d\mu = \inf_n \frac{1}{n} \int \log \|\phi^n\| d\mu.$$

If $\log^+ \|\theta^{-1}\| \in L^1(\mu)$ *then*

$$\lambda_{\min}(x) = \lim_n -\frac{1}{n} \log \|\phi^n(x)^{-1}\|$$

exist for μ-almost everywhere $x \in X$. Moreover, $\lambda_{\min}$ is μ-integrable, with

$$\int \lambda_{\min} d\mu = \lim_n -\frac{1}{n} \int \log \|(\phi^n)^{-1}\| d\mu = \sup_n -\frac{1}{n} \int \log \|(\phi^n)^{-1}\| d\mu.$$

Proof. As $\|B_1 B_2\| \leq \|B_1\| \, \|B_2\|$ for every $B_1, B_2 \in \mathrm{GL}(d)$, we get from Remark 4.1.2 that $(\log \|\phi^n\|)_n$ is a subadditive sequence. Since $\log^+ \left\|\theta^{\pm 1}\right\| \in L^1(\mu)$, all the conditions of Theorem 3.1.3 are satisfied and the conclusion follows. $\square$

Remark 4.1.4. If in addition, we assume that the measure preserving system $(X, \mathcal{A}, \mu, T)$ is ergodic, then Furstenberg's Theorem can also strenghtened and the conclusion would be that $\lambda_{\max} = \lim_n \frac{1}{n} \int \log \|\phi^n\| d\mu$ and $\lambda_{\min} = \lim_n -\frac{1}{n} \int \log \|(\phi^n)^{-1}\| d\mu$.

The multiplicative ergodic theorem of Oseledets, which we state now, offers a significant refinement to the conclusion of the Furstenberg theorem.

Theorem 4.1.5 (Oseledets). *Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving system. Given any measurable function $\theta : X \to \mathrm{GL}(d)$, if $\log^+ \|\theta^{\pm 1}\| \in L^1(\mu)$ then, for μ-almost every $x \in X$ there exist a positive integer $k = k(x)$ and real numbers $\lambda_1(x) > \cdots > \lambda_k(x)$ and a filtration (that is, a decreasing sequence of vector subspaces)*

$$\mathbb{R}^d = V_x^1 > \cdots > V_x^k > V_x^{k+1} = \{0\}$$

such that, for every $i \in \{1, \ldots, k\}$ and μ-almost every $x \in X$,

(a1) $k(T(x)) = k(x)$ *and* $\lambda_i(T(x)) = \lambda_i(x)$ *and* $\theta(x) \cdot V_x^i = V_{T(x)}^i$*;*

(b1) $\lim_n \frac{1}{n} \log \|\phi^n(x) v\| = \lambda_i(x)$ *for every* $v \in V_x^i \backslash V_x^{i+1}$*;*

(c1) $\lim_n \frac{1}{n} \log |\det \phi^n(x)| = \sum_{i=1}^k d_i(x) \lambda_i(x)$*, where* $d_i(x) = \dim V_x^i - \dim V_x^{i+1}$.

Moreover, the numbers $k(x)$ and $\lambda_1(x), \ldots, \lambda_k(x)$ and the subspaces $V_x^1, \ldots, V_x^k$ depend measurably on the point x.

The numbers $\lambda_i(x)$ are called the Lyapunov exponents of θ at the point x. They satisfy $\lambda_1 = \lambda_{\max}$ and $\lambda_k = \lambda_{\min}$. For this reason, we also call $\lambda_{\max}(x)$ and $\lambda_{\min}(x)$ the extremal Lyapunov exponents of θ at the point x. Each $d_i(x)$ is called the multiplicity of the Lyapunov exponent $\lambda_i(x)$.

A proof of Theorem 4.1.5 can be found in Theorem 4.1 from [4].

We introduce now a simplified version of the multiplicative ergodic theorem for 2-dimensional cocycles. This offers a clear glimpse into the essential aspects of the theorem within a transparent context.

Let us denote by A^n, for $n \in \mathbb{N}$, the cocycle defined by A over T, as in Definition 4.1.1.

Theorem 4.1.6 (Oseledets in dimension 2). *Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving system. Consider a measurable function $A : X \to \mathrm{GL}(2)$ satisfying $\log^+ \left\|A^{\pm 1}\right\| \in L^1(\mu)$. Then, for μ-almost every $x \in X$, for every $v \in \mathbb{R}^2$,*

(1) either $\lambda_-(x) = \lambda_+(x)$ *and*

$$\lim_n \frac{1}{n} \log \|A^n(x) v\| = \lambda_{\pm}(x), \text{ for all } v \in \mathbb{R}^2;$$

(2) or $\lambda_+(x) > \lambda_-(x)$ and there exists a vector line $E^s_x \subset \mathbb{R}^2$ such that

$$\lim_n \frac{1}{n} \log \|A^n(x)v\| = \begin{cases} \lambda_-(x) & \text{if } v \in E^s_x \backslash \{0\} \\ \lambda_+(x) & \text{if } v \in \mathbb{R}^2 \backslash E^s_x \end{cases}.$$

Moreover $A(x) \cdot E^s_x = E^s_{T(x)}$ for every x as in (2).

4.2 Proof of Oseledets's Theorem in dimension 2

Define $c(x) = |\det A(x)|^{1/2}$ and then let $B : X \to \mathrm{SL}(2)$ be given by $A(x) = c(x)B(x)$. Observe that $\log c$ and $\log^+ \left\|B^{\pm 1}\right\|$ are in $L^1(\mu)$, and that for μ-almost every $x \in X$,

$$\lim_n \frac{1}{n} \log \|A^n(x)v\| = t(x) + \lim_n \frac{1}{n} \log \|B^n(x)v\| \qquad \text{for every } v \neq 0$$

where $t(x) = \lim_n n^{-1} \sum_{j=0}^{n-1} \log c\left(T^j(x)\right)$ is the Birkhoff time average of the function $\log c$.

Then, the associated cocycles A^n and B^n have the same Oseledets filtration at almost every point. Moreover, the Lyapunov spectrum of the former cocycle is the $t(x)$-translate of the Lyapunov spectrum of the latter.

Therefore, it is enough to prove the theorem in the case A is in $\mathrm{SL}(2)$, which we assume from now.

Consider any x as in the conclusion of Theorem 4.1.3 and let $\lambda(x) = \lambda_+(x) = -\lambda_-(x)$ (using the definitions of λ_{min} and λ_{max} and the fact that A has determinant 1). First, let us consider $x \in X$ such that $\lambda(x) = 0$. For any $v \in \mathbb{R}^2$,

$$\|A^n(x)\|^{-1} \|v\| = \left\|A^n(x)^{-1}\right\|^{-1} \|v\| \leq \|A^n(x)v\| \leq \|A^n(x)\| \, \|v\|,$$

and so

$$\frac{1}{n} \log\left(\|A^n(x)\|^{-1} \|v\|\right) \leq \frac{1}{n} \log \|A^n(x)v\| \leq \frac{1}{n} \log\left(\|A^n(x)\| \, \|v\|\right).$$

Sending $n \to \infty$ the left-hand side goes to $-\lambda(x) = 0$ and the right-hand side goes to $\lambda(x) = 0$. So, we are done with the case when the exponent vanishes.

Lemma 4.2.1. *Given $B \in \mathrm{SL}(2)$ such that $\|B\| \neq 1$, there exist unit vectors s and u such that $\|B(u)\| = \|B\|$ and $\|B(s)\| = \left\|B^{-1}\right\|^{-1} = \|B\|^{-1}$. These vectors are unique, up to multiplication by -1, they are orthogonal, and their images $B(s)$ and $B(u)$ are also orthogonal.*

Proof. Applying the spectral theorem to B^*B, where B^* is the transpose of B, we may consider the orthogonal and unit vectors s and u to be the eigenvectors of B^*B associated to the eigenvalues λ_1 and λ_2. Since $\lambda_1\lambda_2 = det(B^*B) = 1$, we have that $\lambda_2 = \frac{1}{\lambda_1}$. By definition we also have that $B^*Bu = \lambda_1 u$ and $B^*Bs = \frac{1}{\lambda_1}s$. So, $< Bu, Bs >=< u, B^*Bs >=< u, \frac{1}{\lambda_1}s >= 0$. It remains to show that $\|B(u)\| = \|B\|$ and $\|B(s)\| = \left\|B^{-1}\right\|^{-1} = \|B\|^{-1}$. Notice that $\lambda_1 = \|B^*Bu\| \leq \|B\|^2$ and since λ_1 is the maximal eigenvalue, we get $\|B\|^2 = \lambda_1$. On the other part $\|Bu\|^2 =< u, B^*Bu >= \lambda_1$, so we have $\|B(u)\| = \|B\|$. Now $\|Bs\|^2 = \frac{1}{\lambda_1} = \|B\|^{-1}$, and the fact that $\|B\| = \|B^{-1}\|$ comes from the definition of the norm and the non-singularity of B :

$$\|B^{-1}\| = (\inf \frac{\|Bx\|}{\|x\|})^{-1} = \sup \frac{\|Bx\|}{\|x\|} = \|B\|.$$

□

Now let us suppose that $\lambda(x) > 0$. Then $\|A^n(x)\| \approx e^{n\lambda(x)}$ is larger than 1 for every large n. So, by Lemma 4.2.1, there exist unit vectors $s_n(x)$ and $u_n(x)$, respectively, most contracted and most expanded under $A^n(x)$:

$$\|A^n(x)s_n(x)\| = \|A^n(x)\|^{-1} \quad \text{and} \quad \|A^n(x)u_n(x)\| = \|A^n(x)\| . \tag{4.1}$$

Lemma 4.2.2. *The angle $\measuredangle\,(s_n(x), s_{n+1}(x))$ decreases exponentially:*

$$\limsup_n \frac{1}{n} \log |\sin \measuredangle\,(s_n(x), s_{n+1}(x))| \leq -2\lambda(x).$$

Proof. Let us denote $\alpha_n = \measuredangle\,(s_n(x), s_{n+1}(x))$. Then

$$s_n(x) = \sin \alpha_n u_{n+1}(x) + \cos \alpha_n s_{n+1}(x).$$

It follows that

$$\left\|A^{n+1}(x)s_n(x)\right\| \geq \left\|\sin \alpha_n A^{n+1}(x)u_{n+1}(x)\right\| = |\sin \alpha_n| \left\|A^{n+1}(x)\right\| ,$$

and

$$\left\|A^{n+1}(x)s_n(x)\right\| \leq \|A\,(T^n(x))\| \, \|A^n(x)s_n(x)\| = \|A\,(T^n(x))\| \, \|A^n(x)\|^{-1} .$$

Hence

$$|\sin \alpha_n| \leq \frac{\|A\,(T^n(x))\|}{\|A^{n+1}(x)\| \, \|A^n(x)\|}.$$

Lemma 2.2.3 ensures that

$$\lim_n \frac{1}{n} \log \|A\,(T^n(x))\| = 0.$$

So, taking limit superior on both sides of the previous expression, we find that

$$\limsup_n \frac{1}{n} \log |\sin \alpha_n| \leq -2\lambda(x).$$

This completes the proof of the lemma. ☐

The projective space $\mathbb{P}^2$ is the set of equivalence classes of $\mathbb{R}^2\backslash\{0\}$ under the equivalence relation $\sim$ defined by $x \sim y$ if there is a nonzero element $\lambda \in \mathbb{R}$ such that $x = \lambda y$.

Lemma 4.2.3. *The sequence $(s_n(x))_n$ is Cauchy in the projective space.*

Proof. Consider any $\varepsilon > 0$ such that $-2\lambda(x) + \varepsilon < 0$. By Lemma 4.2.2,

$$|\sin \alpha_n| \leq e^{n(-2\lambda(x)+\varepsilon)}$$

for every large n. Then, up to replacing some $s_j(x)$ by $-s_j(x)$,

$$\|s_n(x) - s_{n+1}(x)\| \leq 2e^{n(-2\lambda(x)+\varepsilon)}$$

for every large n. Consequently, there exists $C > 0$ such that

$$\|s_{n+k}(x) - s_n(x)\| \leq Ce^{n(-2\lambda(x)+\varepsilon)}$$

for every $k \geq 1$ and n large enough. In particular, the sequence is Cauchy. Define $s(x) = \lim s_n(x)$ whenever the limit exists. ☐

Lemma 4.2.4. *If* $a_n, b_n > 0$ *for every* n *then*

(1) $\limsup_n \frac{1}{n} \log(a_n + b_n) = \max\left\{\limsup_n \frac{1}{n}\log a_n, \limsup_n \frac{1}{n}\log b_n\right\}$.

(2) $\limsup_n \frac{1}{n} \log\sqrt{a_n^2 + b_n^2} = \max\left\{\limsup_n \frac{1}{n}\log a_n, \limsup_n \frac{1}{n}\log b_n\right\}$.

Proof. $\forall n \in \mathbb{N}, \frac{1}{n}\log a_n \leq \frac{1}{n}\log(a_n + b_n)$. Then, $\limsup_n \frac{1}{n}\log a_n \leq \limsup_n \frac{1}{n}\log(a_n + b_n)$. Similarly, $\limsup_n \frac{1}{n}\log b_n \leq \limsup_n \frac{1}{n}\log(a_n + b_n)$. So we obtain that,

$$\max\left\{\limsup_n \frac{1}{n}\log a_n, \limsup_n \frac{1}{n}\log b_n\right\} \leq \limsup_n \frac{1}{n}\log(a_n + b_n).$$

Now, notice that

$$\frac{1}{n}\log(a_n + b_n) = \frac{1}{n}\log(1 + \frac{b_n}{a_n}) + \frac{1}{n}\log(a_n) = \frac{1}{n}\log(1 + \frac{a_n}{b_n}) + \frac{1}{n}\log(b_n).$$

Since, for each $n \in \mathbb{N}$ either $\log(1 + \frac{b_n}{a_n})$ or $\log(1 + \frac{a_n}{b_n})$ is bounded by $\log(2)$, we may write

$$\begin{aligned}\limsup_n \frac{1}{n}\log(a_n + b_n) &\leq \limsup_n \max\left\{\frac{1}{n}\log a_n, \frac{1}{n}\log b_n\right\} \\ &\leq \max\left\{\limsup_n \frac{1}{n}\log a_n, \limsup_n \frac{1}{n}\log b_n\right\}\end{aligned}$$

To prove *(2)*, we write $\log\sqrt{a_n^2 + b_n^2} = \frac{1}{2}\log(a_n^2 + b_n^2)$ and then use *(1)* to conclude. □

Lemma 4.2.5. *The vector* $s(x)$ *is contracted at the rate* $-\lambda(x)$ *:*

$$\lim_n \frac{1}{n}\log\|A^n(x)s(x)\| = -\lambda(x).$$

Proof. Let $\beta_n = \measuredangle(s(x), s_n(x))$. Then $s(x) = \cos\beta_n s_n(x) + \sin\beta_n u_n(x)$ and so

$$\|A^n(x)s(x)\| \leq |\cos\beta_n| \|A^n(x)s_n(x)\| + |\sin\beta_n| \|A^n(x)u_n(x)\|.$$

Then, by Lemma 4.2.4,

$$\begin{aligned}&\limsup_n \frac{1}{n}\log\|A^n(x)s(x)\| \\ &\leq \max\left\{\limsup_n \frac{1}{n}|\cos\beta_n| \|A^n(x)s_n(x)\|, \limsup_n \frac{1}{n}|\sin\beta_n| \|A^n(x)u_n(x)\|\right\} \\ &\leq \max\left\{\limsup_n \frac{1}{n}\log\|A^n(x)\|^{-1},\right. \\ &\left.\limsup_n \frac{1}{n}\log|\sin\beta_n| + \limsup_{n\to\infty} \frac{1}{n}\|A^n(x)u_n(x)\|\right\} \\ &\leq \max\{-\lambda(x), -2\lambda(x) + \lambda(x)\} = -\lambda(x).\end{aligned}$$

This completes the proof. □

Lemma 4.2.6. *If* $v \in \mathbb{R}^2$ *is not collinear with* $s(x)$ *then*

$$\lim_n \frac{1}{n}\log\|A^n(x)v\| = \lambda(x).$$

Proof. Denote $\gamma_n = \measuredangle(v, s_n(x))$. Then $v = \cos\gamma_n s_n(x) + \sin\gamma_n u_n(x)$ and so

$$\|A^n(x)v\| \geq |\sin\gamma_n| \, \|A^n(x)u_n(x)\| - |\cos\gamma_n| \, \|A^n(x)s_n(x)\|$$

Note that $|\sin\gamma_n|$ is bounded from zero for all large n, because $s_n(x) \to s(x)$ and v is not collinear to $s(x)$. Using also (4.1),

$$\|A^n(x)u_n(x)\| \approx e^{n\lambda(x)} \text{ and } \|A^n(x)s_n(x)\| \approx e^{-n\lambda(x)}.$$

Substituting this in the previous inequality, and taking the limit as $n \to \infty$, we get

$$\liminf_n \frac{1}{n} \log \|A^n(x)v\| \geq \lambda(x)$$

From $\|A^n(x)v\| \leq \|A^n(x)\| \, \|v\|$ we immediately get the opposite inequality:

$$\limsup_n \frac{1}{n} \log \|A^n(x)v\| \leq \lim_n \frac{1}{n} \log \|A^n(x)\| = \lambda(x).$$

The proof of the lemma is complete. □

Lemma 4.2.7. *$A(x)s(x)$ is collinear to $s(T(x))$.*

Proof. By Lemma 4.2.5,

$$\lim_n \frac{1}{n} \log \|A^n(T(x))A(x)s(x)\| = \lim_n \frac{1}{n+1} \log \left\|A^{n+1}(x)s(x)\right\| = -\lambda(x).$$

By Lemma 4.2.6,

$$\lim_n \frac{1}{n} \log \|A^n(T(x))v\| = \lambda(T(x)) = \lambda(x)$$

for every v not collinear to $s(T(x))$. This implies the claim of the lemma. □

Take E^s_x to be the line $\mathbb{R}s(x)$ generated by $s(x)$. Lemmas 4.2.2 through 4.2.7 contain all the claims in Theorem 4.1.6.

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